

ON REAL BISECTIONAL CURVATURE FOR HERMITIAN MANIFOLDS

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ABSTRACT. Motivated by the recent work of Wu and Yau on the ampleness of canonical line bundle for compact Kähler manifolds with negative holomorphic sectional curvature, we introduce a new curvature notion called *real bisectional curvature* for Hermitian manifolds. When the metric is Kähler, this is just the holomorphic sectional curvature H , and when the metric is non-Kähler, it is slightly stronger than H . We classify compact Hermitian manifolds with constant non-zero real bisectional curvature, and also slightly extend Wu-Yau's theorem to the Hermitian case. The underlying reason for the extension is that the Schwarz lemma of Wu-Yau works the same when the target metric is only Hermitian but has nonpositive real bisectional curvature.

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1. INTRODUCTION AND THE STATEMENT OF RESULTS

The study of holomorphic sectional curvature in Kähler geometry has been a classic topic, and it also attracted some attention in recent years, for example, the work of Heier and collaborators on compact Kähler manifolds with positive or negative holomorphic sectional curvature ([8, 9, 10, 1]), and the recent breakthrough of Wu and Yau [18] in which they proved that any projective Kähler manifold with

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negative holomorphic sectional curvature must have ample canonical line bundle. This result was obtained by Heier et. al. earlier under the additional assumption of the Abundance Conjecture.

In [16], Tosatti and the first named author proved that any compact Kähler manifold with nonpositive holomorphic sectional curvature must have nef canonical line bundle, with that in hand, they were able to drop the projectivity assumption in the aforementioned Wu-Yau Theorem. More recently, Diverio and Trapani [6] further generalized the result by assuming that the holomorphic sectional curvature is only *quasi-negative*, namely, nonpositive everywhere and negative somewhere in the manifold. In [19], Wu and Yau give a direct proof of the statement that any compact Kähler manifold with quasi-negative holomorphic sectional curvature must have ample canonical line bundle.

In this direction, the following conjectures are still open:

Conjecture 1.1. Let M^n be a compact complex manifold.

- (a) If M is Kobayashi hyperbolic, then its canonical line bundle K_M is ample.
- (b) If M admits a Hermitian metric with quasi-negative holomorphic sectional curvature, then K_M is ample.
- (c) If M admits a Hermitian metric with negative holomorphic sectional curvature, then K_M is ample.

By Yau's Schwarz Lemma [24], any Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant must be Kobayashi hyperbolic. So (a) implies (c). Clearly, (b) also implies (c).

As an attempt to push the results of [18] to conjecture (b) or (c) above, and also to study holomorphic sectional curvature on Hermitian manifolds in general, we introduce the following curvature term for Hermitian manifolds.

Definition 1.2. Let (M^n, g) be a Hermitian manifold, and denote by R the curvature tensor of the Chern connection. For $p \in M$, let $e = \{e_1, \dots, e_n\}$ be a unitary tangent frame at p , and let $a = \{a_1, \dots, a_n\}$ be non-negative constants with $|a|^2 = a_1^2 + \dots + a_n^2 > 0$. Define the *real bisectional curvature* of g by

$$B_g(e, a) = \frac{1}{|a|^2} \sum_{i,j=1}^n R_{i\bar{i}j\bar{j}} a_i a_j.$$

We will say that a Hermitian manifold (M^n, g) has *positive real bisectional curvature*, denoted by $B_g > 0$, if for any $p \in M$ and any unitary frame e at p , any nonnegative constants $\{a_1, \dots, a_n\}$ (not all of them zero), it holds that $B_g(e, a) > 0$. The term $B_g \geq 0$, $B_g < 0$, or $B_g \leq 0$ are defined similarly. We will say that B_g is *quasi-positive*, if it is non-negative everywhere, and is positive at a point $p \in M$ for all choices of e and a .

Let c be a constant. It is easy to see that, at any $p \in M$ and for any fixed unitary frame e , then $B_g(e', a) > c$ for any choice of unitary frame e' and nonnegative (but not all zero) constants a if and only if

$$(1) \quad \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi_{ij} \xi_{k\ell} > c \operatorname{tr}(\xi^2)$$

for all non-trivial, nonnegative, Hermitian $n \times n$ matrix ξ . The condition $B_g \geq c$, $< c$, or $\leq c$ can be defined similarly.

Recall that the holomorphic sectional curvature in the direction v is defined by $H(v) = R_{v\bar{v}v\bar{v}}/|v|^4$. If we take e so that e_1 is parallel to v , and take $a_1 = 1$, $a_2 = \dots = a_n = 0$, then B becomes $H(v)$. So $B > 0$ (≥ 0 , < 0 , or ≤ 0) implies $H > 0$ (≥ 0 , < 0 , or ≤ 0). Also, if B is quasi-positive or quasi-negative, then so is H . Using Berger's averaging trick, we will show in the next section that

Proposition 1.3. *If (M^n, g) is Kähler, or more generally, Kähler-like, then $B_g > 0$ (≥ 0 , < 0 , or ≤ 0) if and only if $H > 0$ (≥ 0 , < 0 , or ≤ 0). In the non-Kähler case, there are Hermitian metrics g such that $H > 0$ but $B_g \not\geq 0$, and there are also Hermitian metrics g such that $H < 0$ but $B_g \not\leq 0$.*

Note that the term *Kähler-like* means that the Chern curvature tensor obeys all the symmetries of the curvature of Kähler metrics (see [21]). The second part of the above proposition says that, in the non-Kähler case, real bisectional curvature is indeed a stronger curvature condition than holomorphic sectional curvature, and the concept is a natural generalization of holomorphic sectional curvature for Kähler manifolds. On the other hand, the difference between B_g and H is subtle and not very big, as we will see in the next section. For instance, the sign of B_g does not control the sign of any of the three Ricci curvature tensors (see §2).

It is a natural question to ask when will a Hermitian manifold have constant real bisectional curvature. To this end, we have the following:

Theorem 1.4. *Let (M^n, g) be a compact Hermitian manifold whose real bisectional curvature is constantly equal to c .*

- (a) *If $c \neq 0$, then g is Kähler (thus is a complex space form).*
- (b) *If $c = 0$, then g is balanced, with vanishing first, second, and third Ricci tensor, and its Chern curvature satisfies the property $R_{x\bar{y}u\bar{v}} = -R_{u\bar{v}x\bar{y}}$ for any type $(1, 0)$ complex tangent vectors x, y, u, v .*

We would like to propose the following conjecture:

Conjecture 1.5. *Let M^n ($n \geq 3$) be a compact Hermitian manifold with vanishing real bisectional curvature. Then its Chern curvature tensor $R = 0$.*

By Boothby's theorem, compact Hermitian manifolds with vanishing Chern curvature are precisely the compact quotients of complex Lie groups equipped with left invariant metrics.

Besides the constant real bisectional curvature cases, more generally, it would certainly be very interesting to try to understand the class of all compact Hermitian manifolds with positive (or negative) real bisectional curvature. For instance, we could raise the following

Conjecture 1.6. Let (M^n, g) be a compact Hermitian manifold, B_g its real bisectional curvature, and K_M its canonical line bundle.

- (a) If $B_g > 0$, then M is simply-connected.
- (b) If $B_g > 0$, then M is rationally connected.
- (c) If $B_g \leq 0$, then K_M is nef.
- (d) If B_g is quasi-negative, then K_M is ample.

Of course the part (d) here is just a slightly weaker version of Conjecture 1.1, part (b). In this paper, we will focus on the negative/nonpositive real bisectional curvature cases, and the main observation of this article is that the Schwarz lemma of Wu and Yau ([18, Proposition 9]) can be generalized to Hermitian manifolds when the target metric has negative real bisectional curvature. As a consequence, the results of Wu-Yau [18, 19], Tosatti-Yang [16], and Diverio-Trapani [6] can be partially generalized to the Hermitian case. To be more precise, we have the following:

Theorem 1.7. *Let (M, h) be a compact Hermitian manifold with nonpositive real bisectional curvature. If M is Kählerian, then its canonical line bundle is nef.*

Theorem 1.8. *Let (M, h) be a compact Hermitian manifold with quasi-negative real bisectional curvature. If M is Kählerian, then its canonical line bundle is ample.*

Recall that Kählerian means that the manifold admits a Kähler metric, which of course does not have to be h here. It would certainly be highly desirable to drop this assumption, but at this point we have no idea how to achieve that goal. We do observe the following special case, which provides some partial evidence to part (d) of Conjecture 1.6.

Theorem 1.9. *Let M be a compact Hermitian manifold with quasi-negative real bisectional curvature. Let N^n be a compact complex manifold which admits a holomorphic fibration $f : N \rightarrow Z$, where a generic fiber is a compact Kähler manifold with $c_1 = 0$. Then M^n cannot be bimeromorphic to N^n .*

Also, as immediate consequences of the Hermitian version of Wu-Yau's Schwarz Lemma (see §4), we get the following rigidity results:

Theorem 1.10. *Let (M, g) be a Hermitian manifold with nonnegative second Ricci curvature and with bounded Gauduchon 1-form η . Assume that as a Riemannian manifold, it is complete and has Ricci curvature bounded from below. Let (N, h) be a Hermitian manifold with real bisectional curvature bounded from above by a negative constant. Then any holomorphic map from M to N must be constant.*

Theorem 1.11. *Let (M, g) be a compact Hermitian manifold with nonnegative second Ricci curvature, and (N, h) be a Hermitian manifold with nonpositive real bisectional curvature. Then any non-constant holomorphic map from M to N is totally geodesic.*

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2. THE REAL BISECTIONAL CURVATURE OF HERMITIAN MANIFOLDS

Let (M^n, g) be a Hermitian manifold. Under a local holomorphic coordinate system (z_1, \dots, z_n) , the curvature tensor of the Chern connection has components

$$(2) \quad R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}_j}.$$

Let e be a unitary frame of type $(1, 0)$ tangent vectors and a_1, \dots, a_n be non-negative constants with $|a|^2 = a_1^2 + \dots + a_n^2 > 0$. Recall that the real bisectional curvature B_g in the direction of e and a is defined by

$$B_g(e, a) = \frac{1}{|a|^2} \sum_{i,j=1}^n R_{e_i \bar{e}_i e_j \bar{e}_j} a_i a_j.$$

Remark 2.1. Note that the definition of real bisectional curvature is somewhat analogous to the notion of *quadratic orthogonal bisectional curvature* defined in [20], see also [3, 4, 7, 11, 26]. However, the two curvature notions are actually quite different, in the sense that the former is a slight generalization of holomorphic sectional curvature H (and is actually equivalent to H when the metric is Kähler) while the latter is closer to orthogonal bisectional curvature.

For any type $(1, 0)$ tangent vector $v \neq 0$, if we choose e so that e_1 is parallel to v , and choose $a_1 = 1, a_2 = \dots = a_n = 0$, then we get

$$B_g(e, a) = R_{v\bar{v}v\bar{v}}/|v|^4 = H(v).$$

So the holomorphic sectional curvature is part of the real bisectional curvature, and the sign of B guarantees the sign of H .

Conversely, let ω_{FS} be the Fubini-Study metric on \mathbb{P}^{n-1} with unit volume, and let $[w_1 : \dots : w_n]$ be the standard unitary homogeneous coordinate. Then it is well-known that

$$\int_{\mathbb{P}^{n-1}} \frac{w_i \bar{w}_j w_k \bar{w}_\ell}{|w|^4} \omega_{FS}^{n-1} = \frac{\delta_{ij} \delta_{k\ell} + \delta_{i\ell} \delta_{kj}}{n(n+1)},$$

so if we fix a point $p \in M$ and fix any nonnegative constants b_1, \dots, b_n , not all zero, then by considering the integration

$$\sum_{i,j,k,\ell=1}^n \int_{\mathbb{P}^{n-1}} R_{i\bar{j}k\bar{\ell}} \frac{b_i w_i \bar{b}_j \bar{w}_j b_k w_k \bar{b}_\ell \bar{w}_\ell}{|w|^4} \omega_{FS}^{n-1} = \frac{2}{n(n+1)} \sum_{i,k=1}^n (R_{i\bar{i}k\bar{k}} + R_{i\bar{k}k\bar{i}}) b_i^2 b_k^2,$$

we know that, if $H > 0$, then the real bisectional curvature $B_g > 0$ if the metric g is Kähler-like ([21]), as in this case the two curvature terms in the right hand side of the above equality are equal. For a general Hermitian metric g , if $H > 0$, then we know that, for any unitary frame e and any nonnegative constants a_1, \dots, a_n with at least one of them being positive, it holds that

$$(3) \quad \sum_{i,k} (R_{i\bar{i}k\bar{k}} + R_{i\bar{k}k\bar{i}}) a_i a_k > 0.$$

This is an equivalent way to describe $H > 0$. It is analogous to the definition of $B_g > 0$, but not exactly the same (when the metric is not Kähler-like in the sense of [21]). So we have proved the first part of Proposition 1.3.

To see the second part of Proposition 1.3, let us consider the following example:

Example 2.2. On a ball centered at the origin and with small radius $D \subseteq \mathbb{C}^n$ ($n \geq 2$), consider the $U(n)$ -invariant Hermitian metric g defined by

$$g_{i\bar{j}} = (1 + |z|^2) \delta_{ij} + (\varepsilon - 2) \bar{z}_i z_j$$

where $\varepsilon \in (0, 1)$ is a constant and $|z|^2 = |z_1|^2 + \dots + |z_n|^2$, with (z_1, \dots, z_n) the standard Euclidean coordinate in \mathbb{C}^n . We claim that the metric has positive holomorphic sectional curvature, but the real bisectional curvature is not even nonnegative.

At the origin $z = 0$, since all the first order derivatives of g are zero, we get

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} = -\delta_{ij} \delta_{k\ell} + (2 - \varepsilon) \delta_{i\ell} \delta_{kj}.$$

From this, we see that the holomorphic sectional curvature at the origin is constantly $H(v) = 1 - \varepsilon$, which is positive. On the other hand, if $e_\alpha = \sum_i A_{\alpha i} \frac{\partial}{\partial z_i}$ is a unitary frame at $z = 0$, and a_1, \dots, a_n are nonnegative constants, with not all of them zero, then we have

$$B_g(e, a) = \sum_{\alpha, \beta=1}^n R_{e_\alpha \bar{e}_\alpha e_\beta \bar{e}_\beta} a_\alpha a_\beta = -\left(\sum_{\alpha=1}^n a_\alpha\right)^2 + (2 - \varepsilon) \sum_{\alpha=1}^n a_\alpha^2.$$

If we take $a_1 = \dots = a_n = \frac{1}{\sqrt{n}}$, then we get $B_g(e, a) = -n + 2 - \varepsilon < 0$, since $n \geq 2$. So at the origin, hence in a neighborhood of the origin, the metric g has positive holomorphic sectional curvature, but its real bisectional curvature is not even nonnegative.

Note that locally, if we take a metric h that is given by the matrix ${}^t(g_{i\bar{j}})^{-1}$, then the curvature of h is that of g with opposite sign. In particular, if we take the inverse transpose of the metric in Example 2.2, namely, if we let

$$h_{i\bar{j}} = \frac{1}{1 + |z|^2} \delta_{ij} + \frac{2 - \varepsilon}{(1 + |z|^2)(1 - (1 - \varepsilon)|z|^2)} z_i \bar{z}_j,$$

then it would have $H < 0$ near the origin, but $B \not\leq 0$. This completes the proof of Proposition 1.3. \square

Next, we would like to point out that for a Hermitian manifold, although the sign of the holomorphic sectional curvature H does not control that of the real bisectional curvature B , the two are not too far apart from each other. For instance, the sign of B does not control the sign of any of the three Ricci curvature tensors of the Chern connection. We have the following

Example 2.3. Consider a small ball D in \mathbb{C}^2 centered at the origin, equipped with the Hermitian metric g defined by

$$\begin{aligned} g_{1\bar{1}} &= 1 - |z_1|^2 + (1+b)|z_2|^2 \\ g_{2\bar{2}} &= 1 - (1+4b)|z_1|^2 - |z_2|^2 \\ g_{1\bar{2}} &= (1+b)z_2\bar{z}_1 \end{aligned}$$

where (z_1, z_2) is the Euclidean coordinate of \mathbb{C}^2 and $b > 0$ is a constant. We claim that the metric has positive real bisectional curvature, but its first, second, and third Ricci curvature are not nonnegative.

At the origin $z = 0$, $g_{i\bar{j}} = \delta_{ij}$, and $dg = 0$, so the Chern curvature tensor

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j}$$

at $z = 0$ has components

$$R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = 1, \quad R_{1\bar{1}2\bar{2}} = R_{1\bar{2}2\bar{1}} = R_{2\bar{1}1\bar{2}} = -1 - b, \quad R_{2\bar{2}1\bar{1}} = 1 + 4b,$$

and all other components are zero at $z = 0$. Recall that the first, second, and third Ricci tensor of R are defined by

$$\text{Ric}_{i\bar{j}}^{(1)} = \sum_{k,\ell=1}^n g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}, \quad \text{Ric}_{i\bar{j}}^{(2)} = \sum_{k,\ell=1}^n g^{k\bar{\ell}} R_{k\bar{\ell}i\bar{j}}, \quad \text{Ric}_{i\bar{j}}^{(3)} = \sum_{k,\ell=1}^n g^{k\bar{\ell}} R_{i\bar{\ell}k\bar{j}}.$$

At the origin, we have

$$\begin{aligned} \text{Ric}_{1\bar{1}}^{(1)} &= R_{1\bar{1}1\bar{1}} + R_{1\bar{1}2\bar{2}} = -b \\ \text{Ric}_{2\bar{2}}^{(2)} &= R_{1\bar{1}2\bar{2}} + R_{2\bar{2}2\bar{2}} = -b \\ \text{Ric}_{1\bar{1}}^{(3)} &= R_{1\bar{1}1\bar{1}} + R_{1\bar{2}2\bar{1}} = -b. \end{aligned}$$

So none of the three Ricci curvatures is positive at the origin. On the other hand, we claim that the real bisectional curvature of g is positive at (hence near) the origin. That is, we want to show that

$$B = \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi_{ij} \xi_{k\ell} > 0$$

for any non-trivial, non-negative Hermitian matrix $\xi = (\xi_{ij})$. Write $\xi_{11} = x$, $\xi_{22} = y$, and $\xi_{12} = t$, then we have $x \geq 0$, $y \geq 0$, $|t|^2 \leq xy$, and either x or y is positive. At the origin, we have

$$B = x^2 + y^2 + 3bxy - 2(1+b)|t|^2$$

$$\begin{aligned}
&\geq x^2 + y^2 + 3bxy - 2(1+b)xy \\
&= (x-y)^2 + bxy.
\end{aligned}$$

Clearly, $B > 0$ since either x or y will be positive. So the real bisectional curvature of g is positive in a small neighborhood of the origin, yet each of the three Ricci tensors is not even non-negative. This shows that the sign of the real bisectional curvature does not control the sign of any of the three Ricci tensors.

3. MANIFOLDS WITH CONSTANT REAL BISECTIONAL CURVATURE

In this section, we will prove Theorem 1.4 stated in the introduction. Let (M^n, g) be a compact Hermitian manifold with constant real bisectional curvature c , and let e be any unitary frame, then by the definition of real bisectional curvature B_g , we have

$$\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi_{ij} \xi_{k\ell} = c \sum_i \xi_{ii}^2$$

for any non-trivial, nonnegative Hermitian $n \times n$ matrix ξ . This implies that

$$(4) \quad R_{i\bar{j}k\bar{\ell}} + R_{k\bar{\ell}i\bar{j}} = \begin{cases} 2c, & \text{if } i = j = k = \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Next, let us follow the notations of [21] and denote by $\{\varphi_1, \dots, \varphi_n\}$ the coframe of $(1, 0)$ -forms dual to e , with ω the Kähler form of g , and τ the column vector of torsion $(2, 0)$ -forms of the Chern connection. Denote by η the Gauduchon 1-form, then we have

$$\tau_k = \sum_{i,j} T_{ij}^k \varphi_i \wedge \varphi_j, \quad \eta = \sum_j \eta_j \varphi_j = \sum_{i,j} T_{ij}^i \varphi_j.$$

From Lemma 7 of [21], we have

$$(5) \quad 2T_{ij,\bar{\ell}}^k = R_{j\bar{\ell}i\bar{k}} - R_{i\bar{\ell}j\bar{k}}$$

for any indices $1 \leq i, j, k, \ell \leq n$, where the index after the comma stands for covariant differentiation with respect to the Chern connection. By letting $k = i$ and sum over, we get

$$(6) \quad 2\eta_{j,\bar{\ell}} = \sum_k (R_{j\bar{\ell}k\bar{k}} - R_{k\bar{\ell}j\bar{k}}).$$

By formula (15) of [21], we have $\partial(\omega^{n-1}) = -2\eta \wedge \omega^{n-1}$, hence

$$(7) \quad \partial\bar{\partial}(\omega^{n-1}) = 2\bar{\partial}\eta \wedge \omega^{n-1} + 4\eta \wedge \bar{\eta} \wedge \omega^{n-1}.$$

Integrating it over the compact manifold M^n , we get

$$(8) \quad \int_M \left(\sum_i \eta_{i,\bar{i}} \right) \omega^n = 2 \int_M |\eta|^2 \omega^n.$$

From (6), we get

$$\begin{aligned} 2 \sum_i \eta_{i,\bar{i}} &= \sum_{i,k} (R_{i\bar{i}k\bar{k}} - R_{k\bar{i}i\bar{k}}) = \sum_{i \neq k} (R_{i\bar{i}k\bar{k}} - R_{k\bar{i}i\bar{k}}) \\ &= \sum_{i \neq k} (-R_{k\bar{k}i\bar{i}} + R_{i\bar{i}k\bar{k}}) = -2 \sum_i \eta_{i,\bar{i}} \end{aligned}$$

where we used (4) in the first equality of the second line. So $\sum_i \eta_{i,\bar{i}} = 0$ everywhere on M , thus $\eta = 0$ by (8). That is, the Hermitian manifold (M^n, g) is balanced, i.e. $d\omega^{n-1} = 0$.

Next, recall that the first, second, and third Ricci curvature tensor of the Chern curvature tensor R are defined by

$$(9) \quad \text{Ric}_{i\bar{j}}^{(1)} = \sum_k R_{i\bar{j}k\bar{k}}, \quad \text{Ric}_{i\bar{j}}^{(2)} = \sum_k R_{k\bar{k}i\bar{j}}, \quad \text{Ric}_{i\bar{j}}^{(3)} = \sum_k R_{i\bar{k}k\bar{j}}.$$

By the fact that $\eta = 0$ and (6), we know that

$$(10) \quad \sum_k R_{i\bar{j}k\bar{k}} = \sum_k R_{k\bar{j}i\bar{k}}$$

for any $1 \leq i, j \leq n$. So for any $i \neq j$, we have

$$\sum_k R_{i\bar{j}k\bar{k}} = \sum_k R_{k\bar{j}i\bar{k}} = - \sum_k R_{i\bar{k}k\bar{j}} = - \sum_k \overline{R_{k\bar{i}j\bar{k}}} = - \sum_k \overline{R_{j\bar{i}k\bar{k}}}$$

where the first and last equalities are by formula (10), and the second equality is by (4). This means that $\text{Ric}_{i\bar{j}}^{(1)} = 0$ for all $i \neq j$. Similarly, for any fixed i , since $R_{i\bar{i}i\bar{i}} = c$ by (4), we get

$$\begin{aligned} \sum_k R_{i\bar{i}k\bar{k}} &= \sum_k R_{k\bar{i}i\bar{k}} = c + \sum_{k \neq i} R_{k\bar{i}i\bar{k}} = c - \sum_{k \neq i} R_{i\bar{k}k\bar{i}} \\ &= c - \sum_{k \neq i} \overline{R_{k\bar{i}i\bar{k}}} = 2c - \sum_k \overline{R_{k\bar{i}i\bar{k}}} = 2c - \sum_k \overline{R_{i\bar{i}k\bar{k}}} \end{aligned}$$

where the last equality on the first line is again by (4). So $\text{Ric}_{i\bar{i}}^{(1)} = c$ for each $1 \leq i \leq n$. Thus we have shown that, under the assumption that the real bisectional curvature is constant, and the manifold is compact, then we have

$$(11) \quad c \omega = \text{Ric}^{(1)} = -\sqrt{-1} \partial \bar{\partial} \log \omega^n.$$

So when $c \neq 0$, ω is closed, that is, the metric g is Kähler.

When $c = 0$, namely, when g has vanishing real bisectional curvature, we already see that g is balanced and the first Ricci is zero. Also, by (4), we know that the Chern curvature tensor satisfies the skew-symmetry

$$(12) \quad R_{x\bar{y}u\bar{v}} = -R_{u\bar{v}x\bar{y}}$$

for any type $(1,0)$ tangent vectors x, y, u , and v . By (10), we know that $\text{Ric}^{(3)} = 0$. By (12), we get that

$$\text{Ric}^{(2)} = -\text{Ric}^{(1)} = 0.$$

This completes the proof of Theorem 1.4 stated in §1. \square

Remark 3.1. (a) For the computations in this section, see also [12, Corollary 4.2, Corollary 4.5] and [22, Theorem 3.1].
 (b) We conjecture that a compact Hermitian manifold with vanishing real bisectional curvature must have vanishing Chern curvature, thus are compact quotients of complex Lie groups equipped with left invariant metrics.

4. THE HERMITIAN FORM OF WU-YAU'S SCHWARZ LEMMA

The following formula is known as the Schwarz calculation (e.g. [13], [24]), and we include a slightly simpler proof here for the readers' convenience.

Lemma 4.1. *Let $f : (M, g) \rightarrow (N, h)$ be a holomorphic map between Hermitian manifolds. Then in the local holomorphic coordinates $\{z_i\}$ and $\{w_\alpha\}$ on M and N , respectively, we have the identity*

$$(13) \quad \square_g u = |\nabla df|^2 + \left(g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}^g \right) g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R_{\alpha\bar{\beta}\gamma\bar{\delta}}^h \left(g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} \right) \left(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \right).$$

where $u = \text{tr}_{\omega_g}(f^* \omega_h)$, $f_i^\alpha = \frac{\partial f_\alpha}{\partial z_i}$, where the map f is represented by $w_\alpha = f_\alpha(z)$ locally, ∇ is the induced connection on the bundle $E = T_M^* \otimes f^*(T_N)$, and $\square_g u = \text{tr}_{\omega_g}(\sqrt{-1} \partial \bar{\partial} u)$ is the complex Laplacian of u .

Proof. Let $s = \partial f = f_i^\alpha dz_i \otimes e_\alpha \in \Gamma(M, E)$, where $e_\alpha = f^* \frac{\partial}{\partial w_\alpha}$. Since f is a holomorphic map, s is a holomorphic section of E , i.e. $\bar{\partial} s = 0$. Thus by Bochner's formula, we have

$$\partial \bar{\partial} |s|^2 = \langle \nabla' s, \nabla' s \rangle - \langle \Theta^E s, s \rangle$$

where Θ^E is the curvature of the vector bundle E with respect to the induced metric. More precisely, we have

$$\Theta^E = \Theta_M^* \otimes \text{Id}_{f^*(T_N)} + \text{Id}_{T_M^*} \otimes f^*(\Theta^{T_N}).$$

By taking trace, we obtain

$$\text{tr}_{\omega_g}(\sqrt{-1} \partial \bar{\partial} |s|^2) = |\nabla' s|^2 - \langle \text{tr}_{\omega_g} \sqrt{-1} \Theta^E s, s \rangle$$

which is exactly the formula (13) since $|s|^2 = \text{tr}_{\omega_g} f^* \omega_h$. \square

Recall that on a Hermitian manifold (M, g) , the curvature term $g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}^g$ in (13) is called the *second (Chern) Ricci curvature* and is denoted by $\text{Ric}^{(2)}$. It is different from the classic (first Chern) Ricci curvature tensor

$$\text{Ric}_{k\bar{\ell}}^{(1)} = \sum_{i,j} g^{i\bar{j}} R_{k\bar{\ell}i\bar{j}}^g = -\frac{\partial^2 \log \det g}{\partial z^k \partial \bar{z}^\ell},$$

although they coincide when g is Kähler. As an application of Lemma 4.1, we get

Lemma 4.2. *Let $f : (M, g) \rightarrow (N, h)$ be a holomorphic map between two Hermitian manifolds. Then outside the set of critical points of f , one has*

$$(14) \quad \square_g \log u \geq \frac{1}{u} \left[R_{k\bar{\ell}}^{(2)} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R_{\alpha\bar{\beta}\gamma\bar{\delta}}^h \left(g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} \right) \left(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \right) \right]$$

where $u = \text{tr}_{\omega_g}(f^* \omega_h)$, \square_g is the complex Laplacian, and $R_{k\bar{\ell}}^{(2)} = \text{Ric}_{k\bar{\ell}}^{(2)}$ is the second Ricci curvature of the Hermitian manifold (M, g) .

Proof. By using $|df|^2 = |\partial f|^2 = \text{tr}_{\omega_g} f^* \omega_h$ and the formula (13), we know that if $df \neq 0$,

$$\begin{aligned} \square_g \log |df|^2 &= \frac{\square_g |df|^2}{|df|^2} - \frac{|\partial |df|^2|^2}{|df|^4} \\ &= \frac{\square_g |df|^2}{|df|^2} - \frac{4|\partial |df||^2}{|df|^2} \\ &= \frac{\square_g |df|^2}{|df|^2} - \frac{|\nabla |df||^2}{|df|^2}. \end{aligned}$$

By formula (13), we have

$$\begin{aligned} \square_g \log |df|^2 &= \frac{R_{k\bar{\ell}}^{(2)} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R_{\alpha\bar{\beta}\gamma\bar{\delta}}^h \left(g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} \right) \left(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \right)}{|df|^2} \\ &\quad + \frac{|\nabla |df|^2 - |\nabla |df||^2}{|df|^2}. \end{aligned}$$

The well-known Kato's inequality (e.g. [2]) says that for any section ξ of an abstract Riemannian vector bundle (E, ∇) , one has

$$|\nabla |\xi|| \leq |\nabla \xi|,$$

outside the zero set of ξ . Hence, we get (14). \square

By using formula (14), we obtain the following refined version of Yau's Schwarz calculation on Hermitian manifolds, which is also analogous to Royden's formulation ([14]) (see also [17],[18]).

Theorem 4.3. *Let $f : (M, g) \rightarrow (N, h)$ be non-constant holomorphic map between two Hermitian manifolds. Suppose that the second Ricci curvature of g satisfies*

$$(15) \quad \text{Ric}^{(2)}(g) \geq -\lambda \omega_g + \mu f^* \omega_h$$

for continuous functions λ, μ where $\mu \geq 0$, and the real bisectional curvature of h is bounded from above by a continuous function $-\kappa \leq 0$ on N . Then we have

$$(16) \quad \square_g u \geq -\lambda u + \left(\frac{f^* \kappa}{r} + \frac{\mu}{n} \right) u^2$$

and outside the zero locus of df :

$$(17) \quad \square_g \log u \geq -\lambda + \left(\frac{f^* \kappa}{r} + \frac{\mu}{n} \right) u$$

where $u = |df|^2 = \text{tr}_{\omega_g}(f^*\omega_h)$ and r is the maximal rank of df .

Proof. By formula (15), we have

$$\begin{aligned} R_{k\bar{\ell}}^{(2)} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} &\geq (-\lambda g_{k\bar{\ell}} + \mu h_{\gamma\bar{\delta}} f_k^\gamma \overline{f_\ell^\delta}) g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} \\ (18) \qquad \qquad \qquad &\geq -\lambda (\text{tr}_{\omega_g} f^* \omega_h) + \frac{\mu}{n} (\text{tr}_{\omega_g} f^* \omega_h)^2, \end{aligned}$$

where in the last step we used the fact $\mu \geq 0$ and the Cauchy-Schwarz inequality:

$$(19) \qquad (h_{\gamma\bar{\delta}} f_k^\gamma \overline{f_\ell^\delta}) g^{k\bar{q}} g^{p\bar{\ell}} (h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta}) \geq \frac{1}{n} (\text{tr}_{\omega_g} f^* \omega_h)^2.$$

Indeed, if we set $g_{i\bar{j}} = \delta_{ij}$ and denote by $\Phi_{p\bar{q}} = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta}$, then (19) is equivalent to

$$(20) \qquad \sum_{p,q} |\Phi_{p\bar{q}}|^2 \geq \frac{1}{n} \left(\sum_p \Phi_{p\bar{p}} \right)^2,$$

which is obviously true. Next, we use ideas in [14] to estimate the second term on the right hand side of (14).

At a fixed point $p \in M$, by taking unitary changes of coordinates at p and $f(p)$, we may assume that the matrix $[f_i^\alpha]$ has the canonical form

$$(21) \qquad f_i^\alpha = \lambda_i \delta_i^\alpha$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \lambda_{r+1} = \dots = 0$, where r is the rank of the matrix $[f_i^\alpha]$. Then

$$\text{tr}_{\omega_g} f^* \omega_h = \sum_i \lambda_i^2 = \sum_\alpha \lambda_\alpha^2.$$

Hence, we have

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}}^h \left(g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} \right) \left(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \right) &= \sum_{\alpha,\beta,\gamma,\delta,i,k} R_{\alpha\bar{\beta}\gamma\bar{\delta}}^h \lambda_i^2 \lambda_k^2 \cdot \delta_i^\alpha \delta_i^\beta \delta_k^\gamma \delta_k^\delta \\ (22) \qquad \qquad \qquad &= \sum_{\alpha,\gamma,i,k} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}^h \lambda_i^2 \lambda_k^2 \cdot \delta_i^\alpha \delta_k^\gamma = \sum_{\alpha,\gamma} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}^h \lambda_\alpha^2 \lambda_\gamma^2. \end{aligned}$$

Since the real bisectional curvature of (N, h) is bounded from above by $-\kappa \leq 0$, we get

$$(23) \qquad \sum_{\alpha,\gamma} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}^h \lambda_\alpha^2 \lambda_\gamma^2 \leq -\kappa \left(\sum_\alpha \lambda_\alpha^2 \right)^2 = -\frac{\kappa}{r} \left(\sum_\alpha \lambda_\alpha^2 \right)^2 = -\frac{\kappa}{r} (\text{tr}_{\omega_g} f^* \omega_h)^2.$$

The last inequality follows from the fact that $\kappa \geq 0$ and r is the maximal number of nonzero elements of λ_α . Therefore, by using formulas (13), (18) and (23), we obtain (16). By using (14), (18) and (23), we obtain (17). \square

If we apply the above theorem to the identity map of M , we get the following:

Corollary 4.4. *Let M be a compact complex manifold with two Hermitian metrics g, h , such that h has real bisectional curvature bounded above by a continuous function $-\kappa \leq 0$ on M , and g satisfies*

$$(24) \quad \text{Ric}^{(2)}(g) \geq -\lambda\omega_g + \mu\omega_h,$$

for some continuous functions λ, μ with $\mu \geq 0$. Then we have

$$(25) \quad \square_g \log u \geq \left(\frac{\kappa}{n} + \frac{\mu}{n} \right) u - \lambda,$$

where $u = \text{tr}_{\omega_g} \omega_h$. In particular, if κ, λ, μ are all constants and $\kappa + \mu > 0$, then

$$(26) \quad \sup_M u \leq \frac{n\lambda}{\kappa + \mu}.$$

As an immediate consequence of the above Schwarz calculation and Yau's generalized maximum principle [23], we get a proof of Theorem 1.10. Since the statement involves general Hermitian manifolds, let us give a detailed proof for the convenience of the readers. First, recall that Yau's maximum principle says that, on a complete Riemannian manifold M with Ricci curvature bounded from below, if v is a smooth function bounded from below, then for any $\varepsilon > 0$, there exists a point p_ε in M , such that at p_ε ,

$$(27) \quad |\nabla v| < \varepsilon, \quad \Delta v > -\varepsilon, \quad v(p_\varepsilon) < \inf v + \varepsilon.$$

Here Δ is the Laplacian and ∇ the Levi-Civita connection on M .

Next, let us recall the relation between the Ricci curvatures of the Riemannian and Chern connections of a Hermitian manifold. Let (M^n, g) be a Hermitian manifold. It is well-known that

$$(28) \quad \Delta v = 2 \square_g v + 2 \sum_{i=1}^n (v_i \bar{\eta}_i + v_{\bar{i}} \eta_i)$$

where $\eta = \sum_i \eta_i \varphi_i$ is the Gauduchon 1-form, $v_i = e_i(v)$, $v_{\bar{i}} = \bar{e}_i(v)$, and e is any unitary frame with φ the dual coframe.

Now let us assume that $|\eta| \leq C$ on M for some constant $C > 0$, and let $u \geq 0$ be a smooth function on M satisfying the inequality

$$\square_g u \geq -bu + au^2$$

for some constants a, b , with $a > 0$. Then we claim that Yau's maximum principle will imply that $u \leq \frac{b}{a}$.

To see this, note that by (28) we have

$$(29) \quad \Delta u \geq -2bu + 2au^2 - 4C|\nabla u|.$$

Let us consider the smooth, positive function $v = (u + 1)^{-\frac{1}{2}}$ on M . We have $v' = -v^3/2$, $v'' = 3v^5/4$, and $\nabla v = v'\nabla u$, so

$$\Delta v = -\frac{v^3}{2} \Delta u + \frac{3v^5}{4} |\nabla u|^2 = -\frac{v^3}{2} \Delta u + \frac{3}{v} |\nabla v|^2,$$

and

$$-\frac{2}{v^3}\Delta v + \frac{6}{v^4}|\nabla v|^2 = \Delta u > -bu + au^2 - 4C|\nabla u|$$

by (29). That is, we always have

$$(30) \quad -bu + au^2 < -\frac{2}{v^3}\Delta v + \frac{6}{v^4}|\nabla v|^2 + \frac{8C}{v^3}|\nabla v|.$$

Now if (M^n, g) as a Riemannian manifold is complete and with Ricci curvature bounded from below, then by Yau's maximum principle, for any $\varepsilon > 0$, there will be $p_\varepsilon \in M$ at which (27) holds, so by (30), at p_ε we have

$$(31) \quad -bu + au^2 < (2 + 8C)\varepsilon(u + 1)^{\frac{3}{2}} + 6\varepsilon^2(u + 1)^2.$$

Since $a > 0$ is a constant, by choosing ε sufficiently small, we know that $\sup u$ must be finite. When $\varepsilon \rightarrow 0$, $u(p_\varepsilon) \rightarrow \sup u$, so the above inequality gives $\sup u \leq \frac{b}{a}$. Applying Theorem 4.3 in the case where $\mu = 0$, both $\lambda = b$ and $\kappa = k > 0$ are constants, we get proved the following

Theorem 4.5. *Let (M^n, g) be a Hermitian manifold with bounded Gauduchon 1-form η , and its second (Chern) Ricci curvature is bounded from below by a constant $-b$, and as a Riemannian manifold it is complete and has Ricci curvature bounded from below. Let (N^m, h) be a Hermitian manifold whose real bisectional curvature is bounded from above by a negative constant $-k < 0$. If $f : M \rightarrow N$ is any non-constant holomorphic map, then we must have $b > 0$, and*

$$(32) \quad \sup_M \text{tr}_{\omega_g} f^* \omega_h \leq \frac{rb}{k}$$

where r is the maximum rank of df .

In particular, if we start with $b = 0$ at the beginning, then we know that f must be constant. This gives a proof to Theorem 1.10.

Remark 4.6. There are also some Schwarz type inequalities in [15] under different curvature assumptions.

5. NONPOSITIVE AND QUASI-NEGATIVE REAL BISECTIONAL CURVATURE

In this section we will give proofs to Theorems 1.7 through 1.9 and 1.11 stated in the introduction. The proofs are basically the same as those given in [18], [16], and [6], with the simple fact that, all metrics are equivalent on a compact manifold.

Proof of Theorem 1.7: Let (M^n, h) be a compact Hermitian manifold with non-positive real bisectional curvature B_h . Let g be a Kähler metric on M . Assuming that the canonical line bundle K_M is not nef, and we want to derive a contradiction as in the proof of Theorem 1.1 in [16].

Following [16], first of all, since K_M is not nef, there will be $\varepsilon_0 > 0$ such that $\varepsilon_0[\omega_g] - c_1(M)$ is nef but not a Kähler class. Then for any $\varepsilon > 0$, the class $(\varepsilon_0 + \varepsilon)[\omega_g] - c_1(M)$ is a Kähler class. Write ω for ω_g , and denote by $\text{Ric}(g) =$

$-\sqrt{-1}\partial\bar{\partial}\log\omega^n$ the $(1,1)$ form of the first Chern Ricci of the g , which represents $c_1(M)$, thus there exists smooth function φ_ε and ψ_ε on M such that the Kähler metric

$$\omega_\varepsilon := (\varepsilon_0 + \varepsilon)\omega - \text{Ric}(g) + \sqrt{-1}\partial\bar{\partial}u_\varepsilon,$$

where $u_\varepsilon = \varphi_\varepsilon + \psi_\varepsilon$, satisfies $\omega_\varepsilon^n = e^{u_\varepsilon}\omega^n$, or equivalently,

$$\text{Ric}(\omega_\varepsilon) = \text{Ric}(g) - \sqrt{-1}\partial\bar{\partial}u_\varepsilon = -\omega_\varepsilon + (\varepsilon_0 + \varepsilon)\omega.$$

Since M is compact, there exists a constant $D > 0$ such that $\frac{1}{D}\omega \leq \omega_h \leq D\omega$. So one has

$$\text{Ric}(\omega_\varepsilon) \geq -\omega_\varepsilon + \frac{\varepsilon_0 + \varepsilon}{D}\omega_h.$$

Now if we apply the Schwarz Lemma Corollary 4.4 to the identity map from $(M^n, \omega_\varepsilon)$ onto (M^n, h) , with $\lambda = 1$, $\mu = \frac{\varepsilon_0 + \varepsilon}{D}$, and $\kappa = 0$, then we get

$$\text{tr}_{\omega_\varepsilon}\omega_h \leq \frac{nD}{\varepsilon_0 + \varepsilon}$$

as in [16]. So $\text{tr}_{\omega_\varepsilon}\omega \leq \frac{nD^2}{\varepsilon_0 + \varepsilon}$. The rest of the argument is identical to that of [16], so Theorem 1.7 holds true. \square

Next, let us prove Theorem 1.8. Again the same proof of [6] can be slightly modified to cover this case.

Proof of Theorem 1.8: Suppose that (M^n, h) be a compact Hermitian manifold with quasi-negative real bisectional curvature B_h . Let g be a Kähler metric on M and write ω for ω_g . The canonical line bundle K_M is nef by Theorem 1.7. As observed in [6], it suffices to show

$$(33) \quad c_1^n(K_M) > 0,$$

as then K_M will be big by a result of Demailly and Paun [5]. Thus M will be Moishezon, hence projective as it is assumed to be Kähler. Now the Kawamata Theorem implies that K_M must be ample since it does not contain any rational curves.

Since K_M is nef, by [18, Proposition 8], for any $\varepsilon > 0$, there exists smooth function u_ε on M such that the Kähler metric

$$\omega_\varepsilon := \varepsilon\omega - \text{Ric}(g) + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$$

satisfies $\omega_\varepsilon^n = e^{u_\varepsilon}\omega^n$, or equivalently,

$$(34) \quad \text{Ric}(\omega_\varepsilon) = \text{Ric}(g) - \sqrt{-1}\partial\bar{\partial}u_\varepsilon = -\omega_\varepsilon + \varepsilon\omega,$$

and $u_\varepsilon \leq C$ for such constant C independent of ε . The same proof of the inequality $c_1^n(K_M) > 0$ in [6] will go through provided that, for each $\varepsilon > 0$, there will be smooth function $S_\varepsilon > 0$ on M such that

$$(35) \quad \Delta_{\omega_\varepsilon} \log S_\varepsilon \geq \frac{n+1}{2n}\kappa S_\varepsilon - 1$$

holds, where κ is a continuous function on M which is quasi-positive, namely, non-negative everywhere and positive somewhere.

Now if we consider the Hermitian metric h on M with quasi-negative real bisectional curvature. We may let κ be a smooth quasi-positive function on M such that $B_h \leq -\kappa$. Then by applying Corollary 4.4, which is the Hermitian version of [18, Proposition 9], to the identity map from (M, ω_ε) onto (M, h) , we get the above inequality for the function $S_\varepsilon = \text{tr}_{\omega_\varepsilon} \omega_h$, since we have

$$\text{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + \varepsilon\omega \geq -\omega_\varepsilon + \frac{\varepsilon}{D}\omega_h,$$

where $D > 0$ is a constant such that $\frac{1}{D}\omega \leq \omega_h \leq D\omega$ as before. This completes the proof of Theorem 1.8. \square

Next, let us prove Theorem 1.9, which is a rather special case, but without the assumption that M is Kählerian a priori.

Proof of Theorem 1.9: Let M and N be as in Theorem 1.9. Assume the contrary that there is a bimeromorphic map f from N into M . Since M is compact and with nonpositive holomorphic sectional curvature, by the result of Shiffman and Griffiths, M obeys the Hartog's phenomenon. So any meromorphic map into M must be holomorphic. Let $U \subseteq M$ be the open set where real bisectional curvature is negative. Let Y be a generic fiber of N such that the restriction map $f|_Y : Y \rightarrow M$ is non-constant and its image intersects U . By assumption, Y is a compact Kähler manifold with $c_1 = 0$, thus admits a Ricci-flat Kähler metric by Yau's solution to the Calabi conjecture. By applying the Schwarz Lemma Theorem 4.5 to $f|_Y$, we get a contradiction. So M cannot be bimeromorphic to N . \square

Finally, we prove Theorem 1.11, which is a direct consequence of Lemma 4.1.

Proof of Theorem 1.11: Let $f : (M, g) \rightarrow (N, h)$ be a non-constant holomorphic map between Hermitian manifolds, with M being compact. By the curvature assumptions on M and N , and Lemma 4.1, we get

$$\square_g u \geq |\nabla df|^2$$

where $u = |df|^2 = \text{tr}_{\omega_g} f^* \omega_h$. By Gauduchon's theorem, there exists a smooth function v on M such that $\partial\bar{\partial}(e^v \omega_g^{n-1}) = 0$. So we get

$$\int_M |\nabla df|^2 e^v \omega_g^n \leq \int_M \square_g u e^v \omega_g^n = \int_M n\sqrt{-1}\partial\bar{\partial}u \wedge e^v \omega_g^{n-1} = 0,$$

thus $\nabla df = 0$ everywhere on M . That is, f is totally geodesic. This completes the proof of Theorem 1.11. \square

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